DISTRIBUTIONS OF HOOK LENGTHS IN INTEGER PARTITIONS

MICHAEL GRIFFIN, KEN ONO, AND WEI-LUN TSAI

In memory of Christine Bessenrodt

ABSTRACT. Motivated by the many roles that hook lengths play in mathematics, we study the distribution of the number of t-hooks in the partitions of n. We prove that the limiting distribution is normal with mean $\mu_t(n) \sim \frac{\sqrt{6n}}{\pi} - \frac{t}{2}$ and variance $\sigma_t^2(n) \sim \frac{(\pi^2-6)\sqrt{6n}}{2\pi^3}$. Furthermore, we prove that the distribution of the number of hook lengths that are multiples of a fixed $t \geq 4$ in partitions of n converge to a shifted Gamma distribution with parameter $k = (t-1)/2$ and scale $\theta = \sqrt{2/(t-1)}$.

1. INTRODUCTION AND STATEMENT OF RESULTS

A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of n, denoted $\lambda \vdash n$, is a nonincreasing sequence of positive integers that sum to n. Its *Young diagram* is the left-justified array of boxes where the row lengths are the parts. The *hook* $H(k, j)$ of the cell in position (k, j) is the set of cells below or to the right of that cell, including the cell itself, and the *hook length* $h(k, j) := (\lambda_k - k) + (\lambda'_j - j) + 1$, is the number of cells in the hook $H(k, j)$. Here λ'_j is the number of boxes in column j, which is the same as the number of parts of the partition that are at least j.

7	5	4	3	1
5	3	2	1	
1				

FIGURE 1. Hook lengths for $\lambda = (5, 4, 1)$

Multisets $\mathcal{H}(\lambda)$ of partition hook lengths have many roles in combinatorics, number theory, and representation theory (e.g. [1, 2, 3]). For instance, a standard Young tableaux for a partition λ of n is obtained by writing the numbers 1 through n in the boxes of the Young diagram so that each column and each row forms an increasing sequence. The Frame-Robinson-Thrall hook length formula

$$d_\lambda = \frac{n!}{\prod_{h \in \mathcal{H}(\lambda)} h}$$

gives the number of standard Young tableaux for λ. This is also the degree of the canonical irreducible representation of the symmetric group S_n associated to λ. As another important example, we have the famous Nekrasov-Okounkov identity (see (6.12) of [4])[1]

$$(1.1) \qquad \sum_\lambda q^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z}{h^2}\right) = \prod_{n=1}^{\infty} (1 - q^n)^{z-1},$$

which arises in combinatorics, mathematical physics and the theory of modular forms.

[1]This formula was also obtained by Westbury (see Proposition 6.1 and 6.2 of [5]).

In this paper, we study the numbers $Y_t(n)$ which count the t-hooks (i.e. hooks of length t) among all partitions of n. For fixed t, we derive the limiting behavior of the sequence $\{Y_t(n)\}$ for $n \in \mathbb{N}$, and we give asymptotics for the accumulation function

$$(1.2) \qquad D_t(k;n) := \frac{\#\{\lambda \vdash n \text{ with } \leq k \text{ many hook lengths of size } t\}}{p(n)}.$$

Theorem 1.1. *If t is a fixed positive integer, then the following are true for the sequence $\{Y_t(n)\}$.*
(1) The sequence is asymptotically normal with mean $\mu_t(n) \sim \frac{\sqrt{6n}}{\pi} - \frac{t}{2}$ and variance $\sigma_t^2(n) \sim \frac{(\pi^2-6)\sqrt{6n}}{2\pi^3}$.
(2) If we let $k_{t,n}(x) := \mu_t(n) + \sigma_t(n)x$, then in terms of Gauss's error function $E(x)$ we have

$$\lim_{n \to +\infty} D_t(k_{t,n}(x);n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}}\, dy =: E(x).$$

Remark. The $t = 1$ case of Theorem 1.1 recovers a result by Brennan, Knopfmacher and Wagner [6] on the distribution of ascents in partitions, as this number equals the number of size 1-hooks.

Example. Theorem 1.1 asserts that the limiting distribution of 2-hooks is a normal distribution with mean $\mu_2(n) \sim \frac{\sqrt{6n}}{\pi} - 1$ and variance $\sigma_2^2(n) \sim \frac{(\pi^2-6)\sqrt{6n}}{2\pi^3}$. For $n = 5000$, we find that

$$\sum_{\lambda \vdash 5000} T^{\#\{2 \in \mathcal{H}(\lambda)\}} = 704T + 9211712T^2 + \cdots + 1805943379138T^{98} + 2T^{99}.$$
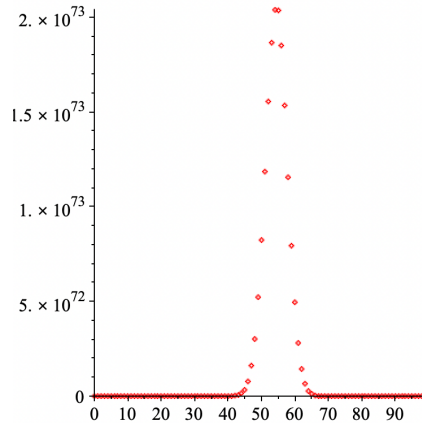
Figure 2 plots $Y_2(5000)$.

FIGURE 2. $Y_2(5000)$

Table 1 illustrates the cumulative distribution approximation $D_2(k_{2,5000}(x);5000) \approx E(x)$.

x	$D_2(k_{2,5000}(x),5000)$	$E(x)$	$D_2(k_{2,5000}(x),5000)/E(x)$
-1.5	$0.0658\ldots$	$0.0668\ldots$	$0.9849\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
0.0	$0.5055\ldots$	$0.5000\ldots$	$1.0011\ldots$
1.0	$0.8246\ldots$	$0.8413\ldots$	$0.9802\ldots$
2.0	$0.9685\ldots$	$0.9772\ldots$	$0.9911\ldots$

TABLE 1. Asymptotics for the cumulative distribution for $n = 5000$

We next consider the sequence $\{\widehat{Y}_t(n)\}$ of distributions of the number of hook lengths in $t\mathbb{N}$ among the partitions of size n. This question is motivated by work of Han that extends (1.1) by giving infinite families of modular forms with level structure and cuspidal divisor. If $\mathcal{H}_t(\lambda)$ is the multiset of hook lengths of λ that are in $t\mathbb{N}$, then he proved (see Theorem 1.3 of [7]) that

$$\sum_\lambda q^{|\lambda|} \prod_{h \in \mathcal{H}_t(\lambda)} \left(y - \frac{tyz}{h^2} \right) = \prod_{n=1}^\infty \frac{(1 - q^{tn})^t}{(1 - (yq^t)^n)^{t-z}(1 - q^n)}.$$

For $t \geq 4$, we prove that the limiting distribution is a shifted Gamma distribution with parameter $k = (t-1)/2$ and scale $\theta = \sqrt{2/(t-1)}$, and we determine asymptotics for the cumulative distribution

$$(1.3) \qquad \widehat{D}_t(k; n) := \frac{\# \{\lambda \vdash n \text{ with } \leq k \text{ many hook lengths in } t\mathbb{N}\}}{p(n)}.$$

Recall (e.g. II.2 of [8]) that a random variable $X_{k,\theta}$ satisfies the *Gamma distribution with parameter $k > 0$ and scale $\theta > 0$* if its probability distribution function is $F_{k,\theta}(x) := \frac{1}{\Gamma(k)\theta^k} \cdot x^{k-1} e^{-\frac{x}{\theta}}$.

Theorem 1.2. *If $t \geq 4$, then the following are true for the sequence $\{\widehat{Y}_t(n)\}$.*
(1) The sequence satisfies

$$\widehat{Y}_t(n) \sim \frac{n}{t} - \frac{\sqrt{3(t-1)n}}{\pi t} \cdot X_{\frac{t-1}{2}, \sqrt{\frac{2}{t-1}}},$$

and has mean $\widehat{\mu}_t(n) \sim \frac{n}{t} - \frac{(t-1)\sqrt{6n}}{2\pi t}$, mode $\widehat{\mathrm{mo}}_t(n) \sim \frac{n}{t} - \frac{(t-3)\sqrt{6n}}{2\pi t}$, and variance $\widehat{\sigma}_t^2(n) \sim \frac{3(t-1)n}{\pi^2 t^2}$.

(2) If we let $\widehat{k}_{t,n}(x) := \widehat{\mu}_t(n) + \widehat{\sigma}_t(n)x$, then in terms of the lower incomplete gamma function we have

$$\lim_{n \to +\infty} \widehat{D}_t(\widehat{k}_{t,n}(x); n) = \frac{\gamma\left(\frac{t-1}{2}; \sqrt{\frac{t-1}{2}}x + \frac{t-1}{2}\right)}{\Gamma\left(\frac{t-1}{2}\right)}.$$

Remark. The proof of Theorem 1.2, which uses properties of Gamma distributions with $k > 1$, does not apply for $t \in \{2, 3\}$ as $(t-1)/2 \leq 1$. Indeed, the $\{\widehat{Y}_2(n)\}$ and $\{\widehat{Y}_3(n)\}$ do not even have continuous limiting distributions. The fact that 100% of n do not have a 2-core or 3-core partition [9] implies that these distributions are populated with many vanishing terms as illustrated by

$$\sum_{\lambda \vdash 19} T^{\#\mathcal{H}_2(\lambda)} = 300T^9 + 185T^8 + 5T^2.$$

Example. Theorem 1.2 gives $\widehat{Y}_{11}(n) \sim \frac{n}{11} - \frac{\sqrt{30n}}{11\pi} \cdot X_{5, \frac{\sqrt{5}}{5}}$, with mean $\widehat{\mu}_{11}(n) \sim \frac{n}{11} - \frac{5\sqrt{6n}}{11\pi}$ and variance $\widehat{\sigma}_{11}^2(n) \sim \frac{30n}{121\pi^2}$. Figure 3 gives $\widehat{Y}_{11}(1000)$.
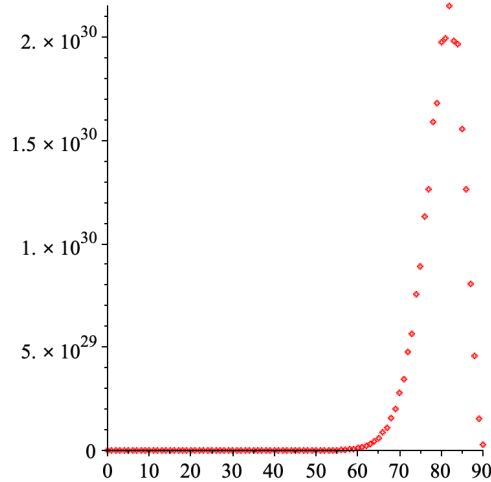
FIGURE 3. $\widehat{Y}_{11}(1000)$

Table 2 illustrates the approximation $\widehat{D}_{11}(k_{11,1000}(x); 1000) \approx \frac{\gamma\left(5; \sqrt{5}x+5\right)}{24} =: \widehat{E}_{11}(x)$.

x	$\widehat{D}_{11}(k_{11,1000}(x); 1000)$	$\widehat{E}_{11}(x)$	$\widehat{D}_{11}(k_{11,1000}(x); 1000)/\widehat{E}_{11}(x)$
-1.00	$0.1319\ldots$	$0.1467\ldots$	$0.8993\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
0.75	$0.7410\ldots$	$0.7954\ldots$	$0.9315\ldots$
1.00	$0.8226\ldots$	$0.8474\ldots$	$0.9707\ldots$
1.25	$0.8872\ldots$	$0.8880\ldots$	$0.9991\ldots$

TABLE 2. Asymptotics for the cumulative distribution for $n = 1000$

This paper is organized as follows. In Section 2 we recall work of Han that offers the relevant enumerative generating functions, and we then determine their asymptotics via the saddle point method, with assistance from the Euler-Maclaurin summation formula. In Section 3 we use these asymptotics to compute the moments of these statistics, which in turn imply Theorems 1.1 and 1.2 thanks to a classical theorem of Curtiss.

ACKNOWLEDGEMENTS

2. NUTS AND BOLTS

We recall work of Han on the enumeration of hook lengths, and we derive important propositions (see Proposition 2.1 and 2.2) that are central to the proof of Theorems 1.1 and 1.2. Han obtained (see Thm. 1.4 and Cor. 5.1 of [7]) the following important generating functions for each fixed $t \geq 1$:

$$(2.1) \quad G_t(T; q) = \sum_{n=0}^{\infty} P_t(n; T)q^n = \sum_{m,n} p_t(m; n)T^m q^n := \sum_{\lambda} q^{|\lambda|} T^{\#\{t \in \mathcal{H}(\lambda)\}} = \prod_{n=1}^{\infty} \frac{(1 + (T-1)q^{tn})^t}{1 - q^n},$$

$$(2.2) \quad \widehat{G}_t(T;q) = \sum_{n=0}^{\infty} \widehat{P}_t(n;T)q^n = \sum_{m,n} \widehat{p}_t(m;n)T^m q^n := \sum_{\lambda} q^{|\lambda|} T^{\#\mathcal{H}_t(\lambda)} = \prod_{n=1}^{\infty} \frac{(1-q^{tn})^t}{(1-(Tq^t)^n)^t(1-q^n)}.$$

The next two propositions on $P_t(n;T)$ and $\widehat{P}_t(n;T)$ are the main results of this section.

Proposition 2.1. *Suppose that* $\eta \in (0,1]$ *and* $\eta \leq T \leq \eta^{-1}$. *If* $c(T) := \sqrt{\pi^2/6 - \mathrm{Li}_2(1-T)}$, *then*

$$P_t(n;T) = \frac{c(T)}{2\sqrt{2}\pi n T^{\frac{t}{2}}} \cdot e^{c(T)\left(2\sqrt{n}-\frac{1}{\sqrt{n}}\right)} \cdot \left(1 + O_\eta(n^{-\frac{1}{7}})\right),$$

where $\mathrm{Li}_2(z) := -\int_0^z \frac{\log(1-u)}{u}du$ *is the dilogarithm function.*

The next proposition is more subtle, and pertains to suitable real sequences.

Proposition 2.2. *If* t *is a positive integer and* $T := \{T_n\}$ *is a positive real sequence for which* $T_n = e^{\frac{\alpha(T)+\varepsilon_T(n)}{\sqrt{n}}}$, *where* $\alpha(T)$ *is real and* $\varepsilon_T(n) = o_T(1)$, *then*

$$\widehat{P}_t(n;T_n) =$$

$$\frac{1}{2^{\frac{7}{4}}3^{\frac{1}{4}}n} \cdot \sqrt{\frac{1}{\sqrt{6}} + \frac{\alpha(T)+\varepsilon_T(n)}{\pi t}} \left(\frac{\pi t}{\pi t + \sqrt{6}\left(\alpha(T)+\varepsilon_T(n)\right)}\right)^{\frac{t}{2}} \cdot e^{\pi\sqrt{n}\left(\sqrt{\frac{2}{3}}+\frac{\alpha(T)+\varepsilon_T(n)}{\pi t}\right)} \cdot (1 + O_T(n^{-\frac{1}{7}})).$$

2.1. **Proof of Proposition 2.1.** The proof of Proposition 2.1 requires the next lemma.

Lemma 2.3. *The following are true.*
If $\eta \in (0,1]$, *then for* $\alpha > 0$ *and* $\eta \leq T \leq \eta^{-1}$ *we have*

$$(2.3) \qquad \sum_{j=1}^{\infty} \log(1-e^{-j\alpha}) = -\frac{\pi^2}{6\alpha} - \frac{1}{2}\log\left(\frac{\alpha}{2\pi}\right) + O(\alpha),$$

$$(2.4) \qquad \sum_{n=1}^{\infty} \frac{t^2 n(T-1)}{T-1+e^{tn\alpha}} = -\frac{\mathrm{Li}_2(1-T)}{\alpha^2} + O_\eta(1),$$

$$(2.5) \qquad \sum_{n=1}^{\infty} \log\left(1+(T-1)e^{-tn\alpha}\right) = -\frac{\mathrm{Li}_2(1-T)}{t\alpha} - \frac{1}{2}\log T + O_\eta(\alpha),$$

$$(2.6) \qquad \sum_{n=1}^{\infty} \frac{t^3 n^2 e^{-tn\alpha}}{(1+(T-1)e^{-tn\alpha})^2} = -\frac{2}{\alpha^3}\frac{\mathrm{Li}_2(1-T)}{T-1} + O_\eta(\alpha).$$

Proof. For $f \in C^{j+1}([a,b])$ and $a,b \in \mathbb{Z}$, Euler-Maclaurin summation (e.g. Thm. 2.1.9 of [10]) gives

$$\sum_{a<n\leq b} f(n) = \int_a^b f(x)dx + \sum_{r=0}^{j} \frac{(-1)^{r+1}}{(r+1)!}\left(f^{(r)}(b) - f^{(r)}(a)\right)B_{r+1} + \frac{(-1)^j}{(j+1)!}\int_a^b B_{j+1}(x-\lfloor x\rfloor)f^{(j+1)}(x)dx,$$

where $B_r(x)$ is the rth Bernoulli polynomial and $B_r := B_r(0)$. Letting $a = 0$ and $j = 0$ gives

$$\sum_{n=1}^{b} \frac{t^2 n(T-1)}{T-1+e^{tn\alpha}} = \int_0^b \frac{t^2(T-1)x}{T-1+e^{t\alpha x}}dx + \frac{t^2(T-1)b}{2(T-1+e^{t\alpha b})}$$

$$+ \int_0^b B_1(x-\lfloor x\rfloor)\frac{t^2(T-1)(T-1+e^{t\alpha x}) - t^3\alpha(T-1)xe^{t\alpha x}}{(T-1+e^{t\alpha x})^2}dx$$

$$= \frac{1}{\alpha^2}\left[\mathrm{Li}_2((1-T)e^{-t\alpha b}) - \mathrm{Li}_2(1-T) - t\alpha\log\left((T-1)e^{-t\alpha b}+1\right)\right] + O_\eta\left(\frac{b^3}{e^{t\alpha b}}\right).$$

To obtain (2.4), we let $b \to \infty$ and find that

$$\sum_{n=1}^{\infty} \frac{t^2 n(T-1)}{T-1+e^{tn\alpha}} = -\frac{\text{Li}_2(1-T)}{\alpha^2} + O_\eta(1).$$

Applying Euler-Maclaurin summation proves the other cases *mutatis mutandis.*   □

Proof of Proposition 2.1. We first note that (2.1) implies that

$$(2.7) \qquad P_t(n;T) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (ze^{ix})^{-n} G_t(T; ze^{ix}) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{g_t(T; ze^{ix})} dx,$$

where $g_t(T; w) := \text{Log}(w^{-n} G_t(T; w))$ for $0 < |w| < 1$. To apply the saddle point method, we must determine $z = e^{-\alpha}$ for $\alpha > 0$, such that $g'_t(T; z) = 0$. By (2.1), this is equivalent to

$$\sum_{j=1}^{\infty} \frac{t^2 j(T-1)}{T-1+e^{tj\alpha}} + \sum_{j=1}^{\infty} \frac{j}{e^{j\alpha}-1} = n.$$

By combining (2.4) with

$$(2.8) \qquad \sum_{j=1}^{\infty} \frac{j}{e^{j\alpha}-1} = \frac{\pi^2}{6\alpha^2} - \frac{1}{2\alpha} + O(1),$$

which holds for $0 < \alpha < 1$, we find that

$$(2.9) \qquad \alpha = c(T) \cdot n^{-\frac{1}{2}} - \frac{1}{4} n^{-1} + O_\eta(n^{-\frac{3}{2}}).$$

We now estimate $g_t(T; z), g''_t(T; z)$, and $g'''_t(T; z)$. Plugging $z = e^{-\alpha}$ into $g_t(T; z)$, we obtain

$$g_t(T; z) = t \sum_{j=1}^{\infty} \log\left(1 + (T-1)e^{-tj\alpha}\right) - \sum_{j=1}^{\infty} \log(1 - e^{-j\alpha}) + n\alpha.$$

Therefore, (2.3), (2.5) and (2.9) gives

$$(2.10) \qquad g_t(T; z) = 2c(T)\sqrt{n} + \frac{1}{2} \log\left(\frac{c(T)}{2\pi T \sqrt{n}}\right) + O_\eta\left(n^{-\frac{1}{2}}\right).$$

Similarly, by using (2.4) and (2.6) we obtain

$$(2.11) \qquad \sum_{j=1}^{\infty} \frac{j^2 e^{-j\alpha}}{(1-e^{-j\alpha})^2} = \frac{\pi^2}{3\alpha^3} - \frac{1}{2\alpha^2} + O(\alpha),$$

which implies that

$$g''_t(T; z) = \left[n + \sum_{j=1}^{\infty} \frac{t^3 j^2 e^{-tj\alpha}}{(1+(T-1)e^{-tj\alpha})^2} - \sum_{j=1}^{\infty} \frac{t^2 j(T-1)}{T-1+e^{tj\alpha}} + \sum_{j=1}^{\infty} \frac{j^2 e^{-j\alpha}}{(1-e^{-j\alpha})^2} - \sum_{j=1}^{\infty} \frac{j}{e^{j\alpha}-1} \right] e^{2\alpha}$$

$$(2.12) \qquad = e^{2c(T)n^{-\frac{1}{2}} + O_\eta(n^{-1})} \left(\frac{2}{c(T)} n^{\frac{3}{2}} + O_\eta(n) \right).$$

By the same argument, we find that

$$(2.13) \qquad g'''_t(T; z) = O_\eta\left(n^2\right).$$

To complete the proof, we now let $P_t(n; T) = I + II$, where

$$I := \frac{1}{2\pi} \int_{|x| \le n^{-5/7}} e^{g_t(T; ze^{ix})} dx \quad \text{and} \quad II := \frac{1}{2\pi} \int_{|x| > n^{-5/7}} e^{g_t(T; ze^{ix})} dx.$$

To estimate I, we use the Taylor expansion of $g_t(T; w)$ centered at the saddle point $z = e^{-\alpha}$

$$g_t(T; w) = g_t(T; z) + \frac{g_t''(T; z)(w - z)^2}{2} + O_\eta(g_t'''(T; z))(w - z)^3).$$

Since $|x| \leq n^{-5/7}$, estimate (2.9) gives

$$w - z = ze^{ix} - z = e^{-\alpha}\left(ix + O\left(x^2\right)\right) = \left(1 + O_\eta\left(n^{-\frac{1}{2}}\right)\right)\left(ix + O\left(n^{-\frac{10}{7}}\right)\right) = ix + O_\eta\left(n^{-\frac{17}{14}}\right).$$

Therefore, we obtain

(2.14) $$g_t(T; w) = g_t(T; z) - \frac{g_t''(T; z)(x)^2}{2} + O_\eta\left(n^{-\frac{1}{7}}\right).$$

Combining (2.10), (2.12), (2.13), and (2.14), we obtain the main term asymptotic

$$I = \frac{e^{g_t(T;z)}}{2\pi}\left[\int_{-\infty}^{\infty} e^{-\frac{g_t''(T;z)x^2}{2}}\, dx - \int_{|x|>n^{-5/7}} e^{-\frac{g_t''(T;z)x^2}{2}}\, dx\right] \cdot \left(1 + O_\eta\left(n^{-\frac{1}{7}}\right)\right)$$

(2.15) $$= \frac{c(T)}{2\sqrt{2\pi}nT^{\frac{t}{2}}} \cdot e^{c(T)\left(2\sqrt{n} - \frac{1}{\sqrt{n}}\right)} \cdot \left(1 + O_\eta\left(n^{-\frac{1}{7}}\right)\right).$$

To estimate the tail error term II, we estimate $\frac{G_t(T;ze^{ix})}{G_t(T;z)}$ using

$$e^{g_t(T;ze^{ix})} = e^{g_t(T;z)}\frac{G_t(T; ze^{ix})}{G_t(T; z)}.$$

Since $T > 0$, letting $w = ze^{ix}$ gives

$$\left|\frac{G_t(T; w)}{G_t(T; z)}\right|^2 \leq \prod_{j=1}^{\infty} \mathrm{Max}\left\{1, \left|\frac{1 + (T - 1)w^j}{1 + (T - 1)z^j}\right|^2\right\}\left|\frac{1 - z^j}{1 - w^j}\right|^2$$

$$\leq \prod_{j=1}^{\infty} \mathrm{Max}\left\{1, \left(1 + \frac{2z^j(1 - T)(1 - \cos(xj))}{(1 - z^j)^2}\right)\right\}\left(1 + \frac{2z^j(1 - \cos(xj))}{(1 - z^j)^2}\right)^{-1}$$

(2.16) $$\leq \prod_{\sqrt{n} \leq j \leq 2\sqrt{n}} \mathrm{Max}\left\{1, \left(1 + \frac{2z^j(1 - T)(1 - \cos(xj))}{(1 - z^j)^2}\right)\right\}\left(1 + \frac{2z^j(1 - \cos(xj))}{(1 - z^j)^2}\right)^{-1}.$$

To reduce to the finite product in the last line, we used the fact that for all $j \geq 1$ we have

$$\mathrm{Max}\left\{1, \left(1 + \frac{2z^j(1 - T)(1 - \cos(xj))}{(1 - z^j)^2}\right)\right\}\left(1 + \frac{2z^j(1 - \cos(xj))}{(1 - z^j)^2}\right)^{-1} \leq 1.$$

We consider two cases (i.e. $T > 1$, and $T \leq 1$) to estimate (2.16). If $T > 1$, and $j \in [\sqrt{n}, 2\sqrt{n}]$, then by (2.9) we have $2z^j/(1 - z^j)^2 \leq c_\eta$, for some $c_\eta > 0$. This implies that

(2.17) $$\left|\frac{G_t(T; w)}{G_t(T; z)}\right|^2 \leq \prod_{\sqrt{n} \leq j \leq 2\sqrt{n}} (1 + c_\eta(1 - \cos(xj)))^{-1}.$$

A short calculation also shows that (2.17) still holds for $T \leq 1$ by choosing a suitable $c_\eta > 0$.

We divide the range of x into two cases $n^{-5/7} \leq |x| \leq \frac{\pi}{2\sqrt{n}}$, and $\frac{\pi}{2\sqrt{n}} \leq |x| \leq \pi$. For the first case, we can use the inequality $1 - \cos(xj) \geq \frac{2}{\pi^2}(xj)^2$ to estimate (2.17), giving

(2.18) $$\left|\frac{G_t(T; w)}{G_t(T; z)}\right|^2 \leq \prod_{\sqrt{n} \leq j \leq 2\sqrt{n}} \left(1 + \frac{2c_\eta}{\pi^2}(xj)^2\right)^{-1} \ll e^{-c_\eta(x^2 n^{\frac{3}{2}})} \ll e^{-c_\eta n^{\frac{1}{14}}}.$$

In the case where $\frac{\pi}{2\sqrt{n}} \leq |x| \leq \pi$, we count the $j \in [\sqrt{n}, 2\sqrt{n}]$ for which there is an $\ell \in \mathbb{Z}$ with $-n^{-\frac{1}{12}} + 2\ell\pi \leq xj \leq n^{-\frac{1}{12}} + 2\ell\pi$. The total number of such j is $\gg n^{1/2} + O(n^{5/12})$. Hence, we have

$$(2.19) \qquad \left| \frac{G_t(T; w)}{G_t(T; z)} \right|^2 \leq \left(1 + c_\eta (1 - \cos(n^{-\frac{1}{12}})) \right)^{-(n^{\frac{1}{2}} + O(n^{\frac{5}{12}}))} \ll e^{-c_\eta n^{\frac{1}{14}}}.$$

By combining (2.18) and (2.19), we obtain the upper bound for the tail

$$II \ll \frac{1}{2\pi} \int_{|x| > n^{-5/7}} e^{g_t(T;z)} \left| \frac{G_t(T; ze^{ix})}{G_t(T; z)} \right| dx \ll_\eta e^{-\frac{c(T)}{\sqrt{n}} - \frac{c_\eta}{2} \cdot n^{\frac{1}{14}}}.$$

As $P_t(n; T) = I + II$, the proposition follows from this inequality and (2.15). $\qquad \square$

2.2. Proof of Proposition 2.2.

For each positive integer n, (2.2) implies that

$$(2.20) \qquad \widehat{P}_t(n; T_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (ze^{ix})^{-n} \widehat{G}_t(T_n; ze^{ix}) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\widehat{g}_t(T_n; ze^{ix})} dx,$$

where $\widehat{g}_t(T_n; w) := \mathrm{Log}(w^{-n} \widehat{G}_t(T_n; w))$ for $0 < |w| < 1$. We aim to locate the saddle point $z = e^{-\beta_n}$, with $\beta_n > 0$. To this end, we solve

$$-\sum_{j=1}^{\infty} \frac{t^2 j}{e^{tj\beta_n} - 1} + \sum_{j=1}^{\infty} \frac{t^2 j T_n^j}{e^{tj\beta_n} - T_n^j} + \sum_{j=1}^{\infty} \frac{j}{e^{j\beta_n} - 1} = n.$$

By (2.8) and the definition of $\alpha(T)$ and $\varepsilon_T(n)$, we obtain we find that

$$\beta_n = \left(\frac{\pi}{\sqrt{6}} + \frac{\alpha(T) + \varepsilon_T(n)}{t} \right) \cdot n^{-\frac{1}{2}} + O_T\left(n^{-1}\right).$$

Since we have $\varepsilon_T(n) = o_T(1)$, it follows that

$$(2.21) \qquad \beta_n = \left(\frac{\pi}{\sqrt{6}} + \frac{\alpha(T)}{t} \right) \cdot n^{-\frac{1}{2}} + o_T\left(n^{-\frac{1}{2}}\right).$$

We now estimate $\widehat{g}_t(T_n; z)$, $\widehat{g}_t''(T_n; z)$, and $\widehat{g}_t'''(T_n; z)$. Plugging $z = e^{-\beta_n}$ into $\widehat{g}_t(T_n; z)$, we obtain

$$\widehat{g}_t(T_n; z) = t \sum_{j=1}^{\infty} \log\left(1 - e^{-tj\beta_n}\right) - t \sum_{j=1}^{\infty} \log\left(1 - T_n^j e^{-tj\beta_n}\right) - \sum_{j=1}^{\infty} \log\left(1 - e^{-j\beta_n}\right) + n\beta_n.$$

Applying (2.3) to all three terms gives

$$(2.22) \qquad \widehat{g}_t(T_n; z) = \frac{t\pi^2}{6(t\beta_n - \log T_n)} + \frac{1}{2} \log\left(\frac{\beta_n}{2\pi}\right) + \frac{t}{2} \log\left(\frac{t\beta_n - \log T_n}{t\beta_n}\right) + n\beta_n + O_T(\beta_n).$$

Similarly, by using (2.8) and (2.11) we obtain

(2.23)
$$\widehat{g}_t''(T_n; z) = \left[n + \frac{\pi^2 t^3}{3(\beta_n t - \log T_n)^3} + \frac{t-1}{2\beta_n^2} - \frac{(t^3 + \pi^2 t^2)}{2(\beta_n t - \log T_n)^2} + \frac{1-t}{2\beta_n} + \frac{t^2}{2(\beta_n t - \log T_n)} + O_T(\beta_n) \right] e^{2\beta_n}.$$

By the same argument, we find that

$$(2.24) \qquad \widehat{g}_t'''(T_n; z) = O_T\left(\beta_n^{-4}\right).$$

Arguing as in the proof of Proposition 2.1 with (2.21), (2.22), (2.23), and (2.24), we obtain

$$\widehat{P}_t(n; T_n) = \frac{e^{\widehat{g}_t(T_n; z)}}{2\pi} \cdot \int_{-\infty}^{\infty} e^{-\frac{\widehat{g}_t''(T_n; z)x^2}{2}} dx \cdot (1 + O_T(n^{-\frac{1}{7}})) = \frac{e^{\widehat{g}_t(T_n; z)}}{2\pi} \cdot \sqrt{\frac{2\pi}{|\widehat{g}_t''(T_n; z)|}} \cdot \left(1 + O_T\left(n^{-\frac{1}{7}}\right)\right)$$

$$= \frac{1}{2^{\frac{7}{4}} 3^{\frac{1}{4}} n} \cdot \sqrt{\frac{1}{\sqrt{6}} + \frac{\gamma(T) + \alpha_T(n)}{\pi t}} \left(\frac{\pi t}{\pi t + \sqrt{6}\left(\alpha(T) + \varepsilon_T(n)\right)}\right)^{\frac{t}{2}} \cdot e^{\pi\sqrt{n}\left(\sqrt{\frac{2}{3}} + \frac{\alpha(T) + \varepsilon_T(n)}{\pi t}\right)} \cdot \left(1 + O_T\left(n^{-\frac{1}{7}}\right)\right).$$

This completes the proof of the proposition.

3. Proof of Theorems 1.1 and 1.2

We prove Theorems 1.1 and 1.2 using the method of moments, where the crucial device is the following classical theorem of Curtiss.

Theorem 3.1 (Theorem 2 of [11]). *Let $\{X_n\}$ be a sequence of real random variables. Then define the corresponding moment generating function*

$$M_{X_n}(r) := \int_{-\infty}^{\infty} e^{rx} dF_n(x),$$

where $F_n(x)$ is the cumulative distribution function associated with X_n. If the sequence $\{M_{X_n}(r)\}$ converges pointwise on a neighborhood of $r = 0$, then $\{X_n\}$ converges in distribution.

Proof of Theorems 1.1 and 1.2. For each $n \geq 1$, we consider the rth power moment

$$(3.1) \qquad M(Y_t(n); r) := \frac{1}{p(n)} \sum_{m=0}^{\infty} p_t(m; n) \cdot e^{\frac{(m - \mu_t(n))r}{\sigma_t(n)}}.$$

By Curtiss's Theorem, combined with the theory of normal distributions, it suffices to show that

$$(3.2) \qquad \lim_{n \to +\infty} M(Y_t(n); r) = e^{\frac{r^2}{2}}.$$

By evaluating $P_t(n; T)$ at $T = 1$ (i.e. $P_t(n; 1) = p(n)$) and $e^{\frac{r}{\sigma_t(n)}}$, we have

$$M(Y_t(n); r) = \frac{P_t(n; e^{\frac{r}{\sigma_t(n)}})}{p(n)} \cdot e^{-\frac{\mu_t(n)}{\sigma_t(n)}r}.$$

Proposition 2.1 gives

$$(3.3) \qquad M(Y_t(n); r) = \frac{c(e^{\frac{r}{\sigma_t(n)}}) \cdot \left(1 + O_\eta\left(n^{-\frac{1}{7}}\right)\right)}{c(1) \cdot \left(1 + O\left(n^{-\frac{1}{7}}\right)\right)} \cdot e^{-\frac{t}{2\sigma_t(n)}r - \frac{\mu_t(n)}{\sigma_t(n)}r + (2n^{\frac{1}{2}} - n^{-\frac{1}{2}}) \cdot (c(e^{\frac{r}{\sigma_t(n)}}) - c(1))}.$$

Since $e^{\frac{r}{\sigma_t(n)}} > 0$ and $e^{\frac{r}{\sigma_t(n)}} \to 1$, as $n \to \infty$, the implied constant can be chosen to be independent of η. By direct calculation of the dilogarithm function, we find that $c(1) = \pi/\sqrt{6}$, and

$$c(e^{\frac{r}{\sigma_t(n)}}) = \frac{\pi}{\sqrt{6}} + \sqrt{\frac{3}{2}} \frac{1}{\pi} \left(\frac{r}{\sigma_t(n)}\right) + \sqrt{\frac{3}{2}} \frac{(\pi^2 - 6)}{4\pi^3} \left(\frac{r^2}{\sigma_t^2(n)}\right) + O\left(\frac{r^3}{\sigma_t^3(n)}\right).$$

Therefore, (3.3) becomes

$$M(Y_t(n); r) = \left(1 + O_r(n^{-\frac{1}{7}})\right) \cdot e^{\left(-\frac{t}{2} - \mu_t(n) + \frac{\sqrt{6n}}{\pi}\right)\frac{r}{\sigma_t(n)} + \sqrt{6n}\left(\frac{\pi^2 - 6}{4\pi^3}\right)\left(\frac{r^2}{\sigma_t^2(n)}\right) + O_r\left(n^{-\frac{3}{4}}\right)}$$

$$= \left(1 + O_r(n^{-\frac{1}{7}})\right) \cdot e^{\frac{r^2}{2} + o_r(1)}.$$

Letting $n \to +\infty$, we obtain (3.2) confirming Theorem 1.1.

To prove Theorem 1.2, we recall that if $k > 1$ and $r < 1/\theta$, then the moment generating function for the random variable $X_{k,\theta}$ is (for example, see II.2 of [8])

$$M(X_{k,\theta}; r) = \frac{1}{(1 - \theta r)^k}.$$

This distribution has mean $\mu_{k,\theta} = k\theta$, mode $\mathrm{mo}_{k,\theta} = (k-1)\theta$, and variance $\sigma_{k,\theta}^2 = k\theta^2$. If a and b are real, then the shifted Gamma distribution $aX_{k,\theta} + b$ has moment generating function

$$M(aX_{k,\theta} + b; r) = e^{br} \cdot M(X_{k,\theta}, ar) = \frac{e^{br}}{(1 - \theta ar)^k},$$

and has mean $ak\theta + b$, mode $a(k-1)\theta + b$, and variance $a^2 k\theta^2$. We compare $\widehat{Y}_t(n)$ with $aX_{k,\theta} + b$, where $(k, \theta) := \left(\frac{t-1}{2}, \sqrt{\frac{2}{t-1}} \right)$, and $a := -1$ and $b := \sqrt{2(t-1)}/2$. Therefore, we assume that $(t-1)/2 > 1$, which is equivalent to $t \geq 4$.

To apply Curtiss's theorem, we compute the moment generating function as in (3.1), with the claimed mean $\widehat{\mu}_t(n) \sim n/t - (t-1)\sqrt{6n}/2\pi t$, and variance $\widehat{\sigma}_t(n) \sim \sqrt{3(t-1)n}/\pi t$. Applying Proposition 2.2 with $\alpha(T) := \pi t r/\sqrt{3(t-1)}$ and $T_n := e^{\frac{\alpha(T)}{\sqrt{n}}}$, we find that

$$M(\widehat{Y}_t(n); r) = \frac{(2^{\frac{7}{4}} 3^{\frac{1}{4}} n)^{-1} \cdot \sqrt{\frac{1}{\sqrt{6}} + \frac{r}{\sqrt{3(t-1)}}} \cdot \left(1 + \sqrt{\frac{2}{t-1}} r \right)^{-\frac{t}{2}} \cdot \left(1 + O_r(n^{-\frac{1}{7}}) \right)}{(4\sqrt{3} n)^{-1} \cdot \left(1 + O(n^{-\frac{1}{7}}) \right)} \cdot e^{\frac{n}{t\widehat{\sigma}_t(n)} r - \frac{\widehat{\mu}_t(n)}{\widehat{\sigma}_t(n)} r}.$$

$$= \frac{e^{\frac{\sqrt{2(t-1)}}{2} r}}{\left(1 + \sqrt{\frac{2}{t-1}} r \right)^{\frac{t-1}{2}}} \cdot \left(1 + O_r(n^{-\frac{1}{7}}) \right).$$

Therefore, Curtiss's theorem gives $\widehat{Y}_t(n) \sim a\widehat{\sigma}_t(n) X_{k,\theta} + b\widehat{\sigma}_t(n) + \widehat{\mu}_t(n)$, as well as the claimed mean, mode and variance. To obtain claim (2), we recall that if $k > 1$, then the Gamma distribution $X_{k,\theta}$ has cumulative distribution function (e.g. II.2 of [8]) $D_{k,\theta}(x) = \gamma\left(k; \frac{x}{\theta} \right) / \Gamma(k)$, where $\gamma(\alpha; x)$ is the lower incomplete Gamma function.   □

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DEPARTMENT OF MATHEMATICS, BRIGHAM YOUNG UNIVERSITY, PROVO, UT 84602
Email address: mjgriffin@math.byu.edu

Department of Mathematics, University of Virginia, Charlottesville, VA 22904

Email address: ko5wk@virginia.edu

Department of Mathematics, University of Virginia, Charlottesville, VA 22904

Email address: wt8zj@virginia.edu